SYMMETRIC PAIRS AND MOMENT SPACES

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Abstract. For a Lie group G, we seek the right definition of a *moment space* for G. One axiom is clear, involving a closed equivariant three-form. We construct this form for symmetric spaces associated to symmetric pairs (H, G) with an additional structure. Furthermore, we prove a decomposition theorem for these pairs over a compact, connected, and semisimple group.

This monograph details work in progess and is not complete.

0. Introduction

The concept of momentum in a physical system with symmetry is quite classical. The collection of conserved quantities becomes in a more modern viewpoint a map from phase space to the dual of the Lie algebra of the symmetry group. When such a *moment map* exists, much more can be said about the topology of phase space. This is the well-known theory of Hamiltonian G-spaces.

Recently, though, Alekseev, Meinrenken, and Malkin [4] have formalized a theory of "q-Hamiltonian" G-spaces, in which the momentum is G- rather than $\mathfrak{g}^*$-valued. There is even a third example for target of a moment map, which corresponds to a noncompact symmetric space.

This leads us naturally to the question of the number of spaces which can serve as target of a moment map. We provide a recipe which given a symmetric space with structure group G and a special pairing on the Lie algebra of the associated symmetric pair, produces a moment space for G. We also show that for suitable G the only moment spaces which arise in this manner are the known ones and combinations of them. This includes results proven independently by Alekseev, Malkin, Meinrenken, and Kosmann-Schwarzbach ([1] and [2]).

What remains to be shown is that every moment space can be produced in this manner. We hope that this the addition of further conditions to the definition of moment space (corresponding to other aspects of the classical Hamiltonian and q-Hamiltonian momenta), this can be done.

Throughout we will assume G is compact. By "G-space" we will always mean manifold or sometimes orbifold on which G acts. Given a G-space M, to each ξ in the Lie algebra $\mathfrak{g}$ of G, there corresponds a fundamental vector field ξ_M on M, given by

$$\xi_M(x) = \frac{d}{dt} \exp t\xi \cdot x\Big|_{t=0}, \tag{1}$$

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for all $x \in M$. This is a Lie algebra *anti*-homomorphism[1] $\mathfrak{g} \to \mathfrak{X}(M)$; i.e., the assignment $\xi \mapsto \xi_M$ is linear and satisfies, for $\xi, \eta \in \mathfrak{g}$,

$$[\xi_M, \eta_M] = -[\xi, \eta]_M. \tag{2}$$

An extremely useful construct for a G-manifold is the equivariant de Rham cohomology. Note that the de Rham complex $\Omega^*(M)$ of smooth differential forms on M has actions of G and $\mathfrak{g}$. For $g \in G$, we can pull-back by the diffeomorphism induced by g, and for $\xi \in \mathfrak{g}$, we have the Lie derivative and interior product operators corresponding to the vector field ξ_M:

$$\mathcal{L}_\xi = \mathcal{L}_{\xi_M} : \Omega^*(M) \longrightarrow \Omega^*(M);$$
$$\iota_\xi = \iota(\xi_M) \colon \Omega^*(M) \longrightarrow \Omega^{*-1}(M);$$

The symmetric algebra $S(\mathfrak{g}^*)$ has a natural action of G given by the coadjoint action on $\mathfrak{g}^*$. The *equivariant de Rham complex* is given as a graded supercommutative algebra:

$$\Omega^k_G(M) \stackrel{\text{def}}{=} \bigoplus_{2\ell+j=k} \left(\Omega^j(M) \otimes S^\ell(g^*)\right)^G.$$

This complex can also be thought of as equivariant $\Omega^*(M)$-valued polynomials on $\mathfrak{g}$. That is, $\alpha \in \Omega^*_G(M)$ is a map $\mathfrak{g} \to \Omega^*(M)$ which is polynomial in any coordinates on $\mathfrak{g}$ and satisfies

$$\alpha(\operatorname{Ad}_g \xi) = (g^{-1})^* \alpha(\xi),$$

for all $g \in G$ and $\xi \in \mathfrak{g}$. From this viewpoint, the Cartan differential can be written

$$\bigl(d_G \alpha\bigr)(\xi) = d\bigl(\alpha(\xi)\bigr) - \iota_\xi \alpha(\xi). \tag{3}$$

Of course, the cohomology of $(\Omega^*_G(M), d_G)$ is called the equivariant de Rham cohomology of M. See [6] for a thorough treatment of equivariant de Rham theory and how it relates to classical Hamiltonian G-spaces.

1. Definitions and (the) Examples

1.1. Moment Space and Moment Map.

Essentially, what we want a moment space to be is something that can serve as the target of a moment map. In the theory of Hamiltonian and q-Hamiltonian G-spaces, we had notions of

- The differential equation characterising a moment map;
- Reduction of G-spaces;
- Fusion; i.e., a multiplicative structure on the particular class of G-spaces;

and so on. The first of these is the one we intend to axiomatize.

Definition. Let G be a compact Lie group. A *premoment space* for G is a pair $(P, \tilde{\Xi})$, where P is a G-manifold and $\tilde{\Xi}$ is a "natural" closed G-equivariant three-form.

By "natural" we mean that $\tilde{\Xi}$ is functorial with respect to inclusions. Since

$$\Omega^3_G(M) = \Omega^3(M)^G \oplus \left(\Omega^1(M) \otimes \mathfrak{g}^*\right)^G,$$

we can write $\tilde{\Xi}$ as

$$\tilde{\Xi} = \Xi + \tau,$$

[1] Introducing a minus sign in the definition (1) of the fundamental vector field would make this a *bona fide* homomorphism of Lie algebras, but would not aid in calculations or intuition.

where $\Xi \in \Omega^3(P)^G$ is the invariant piece and $\tau\colon \mathfrak{g} \to \Omega^1(P)$ is the equivariant piece. It is a linear map. The condition that $\tilde{\Xi}$ be closed can be written as three equations:

$$d\Xi = 0, \tag{4}$$

and for all $\xi \in \mathfrak{g}$,

$$\iota_\xi \Xi = d\tau(\xi) \tag{5}$$

$$0 = \iota_\xi \tau(\xi) \tag{6}$$

This equivariant form allows us to define a moment map.

Definition. Let M be a G-space and P a premoment space for G. M is called P-*momental* if there exists an invariant two-form $\omega \in \Omega^2(M)^G$ and equivariant map $\Phi\colon M \to P$ such that

$$d_G\omega = -\Phi^*\tilde{\Xi}. \tag{7}$$

M is called P-*Hamiltonian* if in addition, for all $x \in M$,

$$\ker \omega_x = \left\{ \xi_M(x) \;\middle|\; \xi \in \ker \tau_{\Phi(x)}\colon \mathfrak{g} \to T^*_{\Phi(x)}P \right\}. \tag{8}$$

Remarks.

1. The condition (7) can be written less obliquely as
$$d\omega = -\Phi^*\Xi; \tag{9}$$
$$\iota(\xi_M)\omega = \Phi^*\tau(\xi), \tag{10}$$
for all $\xi \in \mathfrak{g}$.
2. For $p \in P$, τ_p is defined to be the linear map $\mathfrak{g} \to T^*_pP$ which takes $\xi \in \mathfrak{g}$ to the evaluation of the one-form $\tau(\xi)$ at the point p. In light of (10), we have that for $p \in P$, the fundamental vector fields of all vectors in the kernel of τ_p must annihilate ω. Thus (8) is a condition of minimal degeneracy.

For a given P, the most immediate candidates for P-momental G-spaces are the orbits of G. These have a natural inclusion map $i\colon \mathcal{O} \to P$. Indeed $\tilde{\Xi}$ induces a two-form on each orbit $\mathcal{O}$. If $p \in \mathcal{O}$, $T_p\mathcal{O}$ is spanned by $\{\, \xi_M(p) \mid \xi \in \mathfrak{g} \,\}$, and we define

$$\omega_{\mathcal{O}}\left(\xi_P(p), \eta_P(p)\right) = \tau(\xi)_p\left(\eta_P(p)\right). \tag{11}$$

By (6), this form is well-defined and alternating, and we immediately see that it satisfies (10). $\omega_{\mathcal{O}}$ is characterised by the property that

$$\iota_\xi \omega_{\mathcal{O}} = i^*\tau(\xi).$$

We claim $\omega_{\mathcal{O}}$ is G-invariant, and this is a consequence of the equivariance of τ. For, given $\xi, \eta \in \mathfrak{g}$, and $g \in G$,

$$\begin{aligned} g^*\omega_{\mathcal{O}}\big|_p\big(\xi_P(p), \eta_P(p)\big) &= \omega_{\mathcal{O}}\big|_{gp}\big(g_* \cdot \xi_P(p), g_* \cdot \eta_P(p)\big) \\ &= \omega_{\mathcal{O}}\big|_{gp}\big((\mathrm{Ad}_g\,\xi)_P(gp), g_* \cdot \eta_P(p)\big) \\ &= \tau(\mathrm{Ad}_g\,\xi)_{gp}\big(g_* \cdot \eta_P(p)\big) \\ &= \big((g^{-1})^*\tau(\xi)\big)_{gp}\big(g_* \cdot \eta_P(p)\big) \\ &= \tau(\xi)_p\big((g^{-1})_* \cdot g_* \cdot \eta_P(p)\big) \\ &= \omega_{\mathcal{O}}\big|_p\big(\xi_P(p), \eta_P(p)\big). \end{aligned}$$

From the Cartan "Magic Formula,"

$$0 = \mathcal{L}_\xi \omega_{\mathcal{O}} = d\iota_\xi \omega_{\mathcal{O}} + i_\xi d\omega_{\mathcal{O}}$$

we must have that

$$\begin{aligned} \iota_\xi d\omega_{\mathcal{O}} &= -d\iota_\xi \omega_{\mathcal{O}} \\ &= -di^*\tau(\xi) \\ &= i^* d\tau(\xi) \\ &= -i^*\iota_\xi \Xi, \end{aligned}$$

which verifies the moment condition (9). We have proved the following:

Proposition 1. *Let P be a premoment space for G. Consider a G-orbit $\mathcal{O} \subset P$ with two-form $\omega_{\mathcal{O}}$ given by (11) and inclusion map i. Then $(\mathcal{O}, \omega_{\mathcal{O}}, i \colon \mathcal{O} \to P)$ is a P-momental G-space.*

Definition. A premoment space P is called a *moment space* for G if all orbits $\mathcal{O} \subset P$ are P-Hamiltonian G-spaces with two-form given by (11) and moment map given by inclusion.

This definition encompasses the heretofore known examples as we shall now see.

1.2. **Examples of Moment Spaces.** The oldest and most well-known example of moment space is the dual $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G. Recall [7] that a G-space M is called *Hamiltonian* (which we for the purposes of this monograph will call $\mathfrak{g}^*$-*Hamiltonian*) if there exists $\omega \in \Omega^2(M)^G$ and a map $\Phi \colon M \to \mathfrak{g}^*$ equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$ such that

$$d\omega = 0; \tag{12}$$

$$\iota(\xi_M)\omega = d\tau(\xi); \tag{13}$$

$$\ker \omega_x = 0 \tag{14}$$

for all $\xi \in \mathfrak{g}$ and $x \in M$. The form ω is otherwise known as symplectic. To see the equivariant three-form, we let $\alpha \colon \mathfrak{g} \to C^\infty(\mathfrak{g}^*)$ be the evaluation map $\alpha(\xi)(\ell) = \langle \ell, \xi \rangle$. This is easily seen to be an equivariant map, so α is an equivariant two-form. $\tilde{\Xi} = \tau$ is just $d_G\alpha$, and therefore completely equivariant (having no invariant part); for $\ell \in \mathfrak{g}^*$ and $\lambda \in T_\ell \mathfrak{g}^*$,

$$\tau(\xi)_\ell(\lambda) = \langle \lambda, \xi \rangle$$

Each τ_ℓ is obviously an isomorphism. Let us denote by $\theta_{\mathfrak{g}^*}$ the $\mathfrak{g}^*$-valued one-form which identifies each $T_\ell \mathfrak{g}^*$ with $\mathfrak{g}^*$. Then we can write

$$\tau(\xi) = \langle \theta_{\mathfrak{g}^*}, \xi \rangle .$$

It is also well-known that coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ are symplectic manifolds and indeed Hamiltonian G-spaces.

The second example of moment space is the Lie group itself, as examined in detail by Alekseev, Meinrenken, and Malkin. Let $\mathfrak{g}$ have an invariant inner product B attached to it (so $\mathfrak{G}$ may be for example the category of compact reductive groups). Let $\theta = \theta_G$ (respectively, $\bar{\theta} = \bar{\theta}_G$) in $\Omega^1(G, \mathfrak{g})$ be the left- (respectively, right-)-invariant Maurer-Cartan form on G. So for $g \in G$,

$$\begin{aligned} \theta_g &= L_{g^{-1}*} \colon T_gG \to T_eG = \mathfrak{g}; \\ \bar{\theta}_g &= R_{g^{-1}*} \colon T_gG \to T_eG = \mathfrak{g}. \end{aligned}$$

In a faithful matrix representation of G (and this is the arena in which it is most convenient to do calculations), $\theta = g^{-1}dg$ and $\bar{\theta} = dg\, g^{-1}$. In this case, there is a closed bi-invariant three-form which is canonical with respect to the pairing B:

$$\Xi = \frac{1}{12}B\big(\theta, [\theta,\theta]\big) = \frac{1}{12}B\big(\bar{\theta}, [\bar{\theta},\bar{\theta}]\big),$$

where by the multiple appearances of θ or $\bar{\theta}$ we mean the skew-symmetrization of the corresponding multilinear map (this is a convention we shall follow throughout). Ξ is closed due to the invariance of B and the *Cartan Structure Equations*

$$d\theta = -\frac{1}{2}[\theta,\theta]; \tag{15}$$

$$d\bar{\theta} = -\frac{1}{2}[\bar{\theta},\bar{\theta}] \tag{16}$$

Let G act on itself by conjugation. Then for $\xi \in \mathfrak{g}$, the fundamental vector field on G corresponding to ξ is

$$\xi_G = \xi_L - \xi_R,$$

where ξ_L is the right-invariant vector field on G generated by the left translation action and *vice versa* for ξ_R. Then

$$\iota(\xi_G)\Xi = \frac{1}{12}\iota(\xi_L)B(\bar{\theta},[\bar{\theta},\bar{\theta}]) - \frac{1}{12}\iota(\xi_R)B(\theta,[\theta,\theta]).$$

By invariance of B, this is

$$\frac{1}{4}B(\xi,[\bar{\theta},\bar{\theta}]) - \frac{1}{4}B(\xi,[\theta,\theta]) = \frac{1}{2}dB(\xi,\theta+\bar{\theta}).$$

So we may write $\tau(\xi) = \frac{1}{2}B(\theta+\bar{\theta},\xi)$ as the equivariant (with respect to the conjugation action) piece of the equivariantly closed three-form $\tilde{\Xi} = \Xi + \tau$ on G. Note that since $\bar{\theta}_g = \mathrm{Ad}_g\,\theta_g$, and since B identifies $\mathfrak{g}^*$ with $\mathfrak{g}$, we have $\ker \tau_g = \ker(1 + \mathrm{Ad}_g)$.

Then following [4], we say a G-space M is *q-Hamiltonian* (henceforth, *G-Hamiltonian*) if there exists $\omega \in \Omega^2(M)^G$ and $\Phi\colon M \to G$ equivariant with respect to the conjugation action of G on itself, such that

$$d\omega = -\Phi^*\Xi; \tag{17}$$

$$\iota(\xi_M)\omega = \frac{1}{2}\Phi^*B(\theta+\bar{\theta},\xi); \tag{18}$$

$$\ker\omega_x = \left\{\, \xi_M(x) \;\middle|\; \xi \in \ker(1+\mathrm{Ad}_{\Phi(x)}) \,\right\}, \tag{19}$$

for all $\xi \in \mathfrak{g}$ and $x \in M$. It is in fact this formulation of the moment and Hamiltonian conditions that motivates the general definition. As in the case of $\mathfrak{g}^*$, the canonical transitive G-Hamiltonian G-spaces are conjugacy classes $\mathcal{C} \subset G$.

A final example arises from complexification. The pairing B extends to a complex-bilinear pairing on the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, which we will call $B^{\mathbb{C}}$. If G is simply connected, there exists a unique simply connected complex Lie group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ containing G as a subgroup. Exponentiating the conjugation automorphism gives us a map $\bar{\cdot}\colon G^{\mathbb{C}} \to G^{\mathbb{C}}$, which singles out G as its fixed-point set. Define

$$P^{\mathbb{C}} = \left\{h \in G^{\mathbb{C}} : \bar{h} = h^{-1}\right\}.$$

$P^{\mathbb{C}}$ is preserved by the conjugation action of G. We can restrict canonical forms on $G^{\mathbb{C}}$ to $P^{\mathbb{C}}$. Let $\theta^{\mathbb{C}}$ and $\bar{\theta}^{\mathbb{C}}$ be the Maurer-Cartan forms on $G^{\mathbb{C}}$. Set $\theta_P = \theta^{\mathbb{C}}\big|_{P^{\mathbb{C}}}$ and likewise define $\bar{\theta}_P$. Set

$$\Xi_P = \frac{1}{12}\operatorname{Im} B^{\mathbb{C}}\left(\bar{\theta}_P, \left[\bar{\theta}_P, \bar{\theta}_P\right]\right)$$

A G-space M is called a *q-Hamiltonian G-space with $P^{\mathbb{C}}$-valued moment map* (henceforth, *$P^{\mathbb{C}}$-Hamiltonian G-space*) if there exists $\omega \in \Omega^2(M)^G$ and $\Phi\colon M \to P^{\mathbb{C}}$ equivariant with respect to the conjugation action of G on P such that

$$d\omega = -\Phi^* \Xi_P; \tag{20}$$

$$\iota(\xi_M)\omega = \frac{1}{2\sqrt{-1}}\Phi^* B(\theta_P + \bar{\theta}_P, \xi); \tag{21}$$

$$\ker \omega_x = 0, \tag{22}$$

for all $\xi \in \mathfrak{g}$ and $x \in M$. It can be shown that the pairing $B(\theta_P + \bar{\theta}_P, \xi)$ is purely imaginary, so the equation (21) makes sense. Then $\tau = \frac{1}{2\sqrt{-1}} B(\theta_P + \bar{\theta}_P, \,\cdot\,)$, and when seeking the kernel of τ_p for $p \in P^{\mathbb{C}}$, we find that all the eigenvalues of Ad_p are positive. Hence τ_p is an isomorphism.

These spaces are introduced in [3] but named in [4]. In both, it is shown that $P^{\mathbb{C}}$-Hamiltonian G-spaces are in one-to-one correspondence with $\mathfrak{g}^*$-Hamiltonian G-spaces, due mainly to the fact that with respect to B there is a canonical diffeomorphism $\mathfrak{g}^* \cong P^{\mathbb{C}}$.

A different perspective on this example is through the category of Poisson-Lie G-spaces, e.g., [5]. In that category, the moment map has as its target the *dual group* G^* of G lying in $G^{\mathbb{C}}$. However, ω is no longer G-invariant, but instead the G-action induces a Poisson map $G \times M \to M$.

A crucial observation of these three moment spaces is that they appear as coset spaces of a larger Lie group containing G as a subgroup. Indeed, for G satisfying all reductivity and connectivity conditions necessary:

1. Let $H_0 = G \rtimes \mathfrak{g}^*$, so
$$(g_1, \ell_1)\cdot(g_2, \ell_2) = (g_1 g_2, \ell_1 + \operatorname{Ad}^*_{g_1} \ell_2).$$
This is the right trivialization of the cotangent group T^*G. The map $j_0\colon H_0 \to \mathfrak{g}^*$, $(g, \ell) \mapsto \ell$ gives an equivariant diffeomorphism between the right coset space[2] H_0/G with left G-action and $\mathfrak{g}^*$ with the coadjoint G-action.
2. Let $H_+ = G \times G$, with G embedded as the diagonal subgroup $\Delta(G)$. The map $j_+\colon G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$ gives an equivariant diffeomorphism of $H_+/\Delta(G)$ with G equipped with its conjugation action.
3. Let $H_- = G^{\mathbb{C}}$, which we have already defined. The map $G^{\mathbb{C}} \to G^{\mathbb{C}}$, $h \mapsto h\bar{h}^{-1}$ is seen to land in $P^{\mathbb{C}}$ and descends to an equivariant diffeomorphism between $G^* = G^{\mathbb{C}}/G$ with its left ("dressing") action and $P^{\mathbb{C}}$ with the conjugation action.

Indeed, when we take the moment forms on $\mathfrak{g}^*$, G, and $P^{\mathbb{C}}$ and pull them back to the respective larger Lie groups, we find a way in which they can all be expressed

[2] If $G \subset H$ are Lie groups, we follow the convention that the *right* coset space refers to the space of orbits for the right action of G on H. That is, H/G is the space of all cosets hG as h ranges through H.

similarly. This leads us to a general recipe for providing moment spaces, which we will describe in the next section.

2. Moment Spaces from Symmetric Pairs

2.1. **Review of symmetric spaces, symmetric paris, and symmetric Lie algebras.**

A symmetric space is essentially a space with reflections through each point. Given such a Riemannian manifold M, we can consider the Lie group H (more precisely, H is the connected component of this group) of isometries of M, and the closed subgroup G of all isometries of M fixing a given point. Then $M \cong H/G$ are G-equivariantly diffeomorphic. This pair of groups is easier to work with than M alone, so we axiomatize the data

Definition ([8])**.** Let H be a Lie group and G a closed subgroup. (H, G) is a *symmetric pair* if there exists an involutive automorphism σ of H (that is, $\sigma^2 = \mathrm{Id}$ while $\sigma \neq \mathrm{Id}$) such that

$$(H^\sigma)_0 \subseteq G \subseteq H^\sigma. \tag{23}$$

Here H^σ is the subgroup of H fixed by σ and $(H^\sigma)_0$ its identity component.

A symmetric pair (H, G) is called *Riemannian* if the image of Ad_G in $\mathfrak{gl}(\mathfrak{h})$ is compact.

Remarks. This definition is quite general, and could use some illumination.

1. Since in our context we are fixing G and considering the different Lie groups H such that (H, G) is a symmetric pair, we can restrict our attention to connect groups G. In this case (23) can be replaced by the more transparent identity $G = H^\sigma$.
2. The symmetric pair (H, G) is Riemannian if in particular G is compact.

Let $\mathfrak{h}$ be the Lie algebra of H and $s = d\sigma_e$. Then $(\mathfrak{h}, s)$ has many linear properties analogous to (H, G). It is in fact an example of the following object.

Definition. Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{R}$ and s an involutive automorphism of $\mathfrak{h}$. The pair $(\mathfrak{h}, s)$ is called a *symmetric Lie algebra*. $(\mathfrak{h}, s)$ is called *orthogonal* if $\mathfrak{g} = \mathfrak{h}^\sigma$ (the fixed point subalgebra of $\mathfrak{h}$) is compactly embedded in $\mathfrak{h}$, and is called *effective* if $\mathfrak{g} \cap \mathfrak{z} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{h}$.

In particular, if (H, G) is a Riemannian symmetric pair, the associated symmetric Lie algebra $(\mathfrak{h}, s)$ is orthogonal. If G is semisimple, then $(\mathfrak{h}, s)$ is effective.

Lemma 1. Let $(\mathfrak{h}, s)$ be a symmetric Lie algebra. Write

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$$

for the decomposition of $\mathfrak{h}$ into the subalgebra $\mathfrak{g}$ fixed by σ and the -1 eigenspace $\mathfrak{p}$ of σ. Then

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}; \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}; \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}.$$

Proof. s is a Lie algebra homorphism. Thus

$$s[\zeta_1, \zeta_2] = [s\zeta_1, s\zeta_2]$$

for all ζ_1, $\zeta_2 \in \mathfrak{h}$. □

Let G be fixed and let H_0, H_+ and H_- be the three Lie groups given in Examples 1–3 above. We will demonstrate that they form symmetric pairs.

1. On $H_0 = G \rtimes \mathfrak{g}^*$, the involution σ_0 is the map $(g, \ell) \mapsto (g, -\ell)$. The corresponding symmetric Lie algebra is $\mathfrak{h}_0 = \mathfrak{g} \rtimes \mathfrak{g}^*$ with Lie bracket and involution
$$[\xi \rtimes \lambda, \eta \rtimes \mu] = [\xi, \eta] \rtimes \left(\operatorname{ad}_\xi^* \mu - \operatorname{ad}_\eta^* \lambda\right)$$
$$s_0(\xi \rtimes \lambda) = \xi \rtimes (-\lambda).$$
The symmetric Lie algebra $\mathfrak{h}_0$ is said to be of the *Euclidean type*.[3]
2. On $H_+ = G \times G$, G is embedded as the diagonal $\Delta(G)$. This subgroup is fixed by the involution $\sigma_+(g_1, g_2) = (g_2, g_1)$. The corresponding symmetric Lie algebra is of course $\mathfrak{h}_+ = \mathfrak{g} \times \mathfrak{g}$ with involution
$$s_+(\xi_1 \times \xi_2) = \xi_2 \times \xi_1,$$
fixing the diagonal subalgebra $\mathfrak{d}(\mathfrak{g})$. This symmetric Lie algebra is of the *compact type*.
3. On $H_- = G^{\mathbb{C}}$, we already know the conjugation automorphism of $\mathfrak{g}^{\mathbb{C}}$ and its exponentiated version. These single out the real forms of $\mathfrak{g}^{\mathbb{C}}$ and $G^{\mathbb{C}}$, respectively.
$$s_-(\xi + \sqrt{-1}\eta) = \xi - \sqrt{-1}\eta.$$
This symmetric Lie algebra is of the *noncompact type*.

2.2. **Legendrian structure.** The symmetric pairs given in the last subsection are distinguished; they have a pairing on the associated symmetric Lie algebra which restricts to a nontrivial pairing between the different eigenspaces of s. We make the notion of this type of pairing precise with a definition.

Definition. Let $\mathfrak{w}$ be a vector space over a field k with involution $s\colon \mathfrak{w} \to \mathfrak{w}$. A *Legendrian form* on $\mathfrak{w}$ is a nondegenerate symmetric bilinear form $\Lambda\colon \mathfrak{w} \times \mathfrak{w} \to k$ with respect to which s is skew-symmetric, i.e., for all $w_1, w_2 \in \mathfrak{w}$,
$$\Lambda(sw_1, w_2) = -\Lambda(w_1, sw_2). \tag{24}$$

Note that since $s^2 = \mathrm{Id}$, we can also write (24) as $\Lambda(sw_1, sw_2) = -\Lambda(w_1, w_2)$.

Definition. Let $(\mathfrak{h}, s)$ be a symmetric Lie Algebra. $\mathfrak{h}$ is *Legendrian* if it admits a Legendrian form which is invariant with respect to the adjoint action of $\mathfrak{h}$ on itself: for all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{h}$,
$$\Lambda(\operatorname{ad}_{\zeta_1} \zeta_2, \zeta_3) = -\Lambda(\zeta_2, \operatorname{ad}_{\zeta_1} \zeta_3), \tag{25}$$
or, using symmetry,
$$\Lambda([\zeta_1, \zeta_2], \zeta_3) = \Lambda(\zeta_1, [\zeta_2, \zeta_3]). \tag{26}$$

Definition. Let (H, G) be a symmetric pair. H is *Legendrian* if the associated symmetric Lie algebra $\mathfrak{h}$ admits a Legendrian form which is invariant with respect to the adjoint action of H on $\mathfrak{h}$. That is, in addition to (24) and (25), we must have, for all $\zeta_1, \zeta_2 \in \mathfrak{h}$ and $g \in G$,
$$\Lambda(\operatorname{Ad}_h \zeta_1, \zeta_2) = \Lambda(\zeta_1, \operatorname{Ad}_{h^{-1}} \zeta_2). \tag{27}$$

[3] We are using the same symbols as Helgason to label these types, but the plus and minus signs are reversed. Under our choice of convention, $+$, $-$, and 0 refer not only to the signs of the sectional curvatures of the corresponding symmetric spaces but to the signs of the eigenvalues in a certain linear transformation of $\mathfrak{g}$ which will appear in the next section. Ultimately, the permutation is due to the fact that in the semisimple case, we will take the *negative* of the Killing form on $\mathfrak{g}$ so as to get a *positive*-definite inner product on $\mathfrak{g}$.

Let $\mathfrak{w}$ be a vector space with involution s, and write $\mathfrak{w} = \mathfrak{v} \oplus \mathfrak{p}$ as its decomposition into the +1 and -1 eigenspaces of s. Then a Legendrian form Λ on $\mathfrak{w}$ is zero when restricted to $\mathfrak{v} \times \mathfrak{v}$ and $\mathfrak{p} \times \mathfrak{p}$. That is, these eigenspaces are *isotropic* with respect to Λ. Since $\mathfrak{v}$ and $\mathfrak{p}$ are algebraically complementary, Λ is a nondegenerate pairing between them, and in particular $\mathfrak{w}$ is even-dimensional, isomorphic to $\mathfrak{v} \oplus \mathfrak{v}^*$. If in addition we are given an inner product on $\mathfrak{v}$, we get a linear map $\mathfrak{v} \to \mathfrak{p}$, which is related to the Legendre transform. Hence the attribution.

Remark. This also seems to be related to the concept of a Drinfeld algebra, which is a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$. However, in that formulation, $\mathfrak{g}^*$ is required to be a subalgebra, whereas we will always have $[\mathfrak{g}^*, \mathfrak{g}^*] \subset \mathfrak{g}$. Only in the least interesting, Euclidean case do these definitions overlap.

Proposition 2. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) $\mathfrak{h}_0 = \mathfrak{g} \rtimes \mathfrak{g}^*$ *has Legendrian structure given by*

$$\Lambda_0(\xi_1 \rtimes \lambda_1, \xi_2 \rtimes \lambda_2) = \langle \xi_1, \lambda_2 \rangle + \langle \xi_2, \lambda_1 \rangle . \tag{28}$$

$(H_0 = G \rtimes \mathfrak{g}^*, G)$ *is a Legendrian symmetric pair.*

(b) *Let $\mathfrak{g}$ have an invariant inner product B. $\mathfrak{h}_+ = \mathfrak{g} \times \mathfrak{g}^*$ has Legendrian structure given by*

$$\Lambda_+(\xi_1 \times \eta_1, \xi_2 \times \eta_2) = \frac{1}{2} \left(B(\xi_1, \xi_2) - B(\eta_1, \eta_2) \right) \tag{29}$$

If G is connected, $(H_+ = G \times G, \Delta(G))$ is a Legendrian symmetric pair.

(c) *Again assume $\mathfrak{g}$ has an inner product B. Then B extends to a $\mathbb{C}$-bilinear inner product on $\mathfrak{h}_- = \mathfrak{g} \otimes \mathbb{C}$. $\mathfrak{h}_-$ has a Legendrian structure given by*

$$\Lambda_-(\zeta_1, \zeta_2) = \operatorname{Im} B^{\mathbb{C}}(\zeta_1, \zeta_2). \tag{30}$$

If G is simply connected, then $(H_- = G^{\mathbb{C}}, G)$ is a Legendrian symmetric space.

Proof. Clear. ◻

2.3. **The equivariant form.** The *Ansatz* here is to show that given a Legendrian symmetric space (H, G), we can construct a moment space for G. This space will in fact be the quotient of H by G. We proceed in a series of steps.

For the remainder of this section (H, G) will be a Legendrian symmetric pair with involution σ. We write $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$ as the decomposition of the Lie algebra of H into the eigenspaces of $s = d\sigma|_e$. The Legendrian structure on $\mathfrak{h}$ will be denoted by Λ.

Let θ be the left-invariant Maurer-Cartan form on $\mathfrak{h}$. Given the involution s, we can decompose θ into its "$\mathfrak{g}$-part" and "$\mathfrak{p}$-part". That is, write

$$\gamma = \frac{1+s}{2}\theta; \qquad \pi = \frac{1-s}{2}\theta;$$

so $\gamma \in \Omega^1(H, \mathfrak{g})$ and $\pi \in \Omega^1(H, \mathfrak{p})$. Let $j \colon H \to P$ be the quotient map.

Proposition 3. *Define for $\xi \in \mathfrak{g}$ a one-form*

$$\beta(\xi)_h = \Lambda(\xi, \operatorname{Ad}_h \pi) \in \Omega^1(H). \tag{31}$$

Then

(a) *$\beta(\xi)$ is basic with respect to the right action of G on H, so there is a unique one-form $\tau(\xi) \in \Omega^1(P)$ such that $j^*\tau(\xi) = \beta(\xi)$.*

(b) *The map $\xi \mapsto \beta(\xi)$ is equivariant with respect to the left action of G on H, so τ is an equivariant three-form on P.*
(c) *We have, for all $\xi \in \mathfrak{g}$,*

$$\iota_\xi \tau(\xi) = 0.$$

Proof. For $h \in H$, let R_h and L_h denote left and right multiplication by h as diffeomorphisms of H. Since $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\theta$ and $\sigma(g) = g$, it follows that $R_g^*\pi = \mathrm{Ad}_{g^{-1}}\pi$. Then

$$\begin{aligned}\left(R_g^*\beta(\xi)\right)_h &= \Lambda(\xi, R_g^* \mathrm{Ad}_h \pi)\\ &= \Lambda(\xi, \mathrm{Ad}_{hg}\mathrm{Ad}_{g^{-1}}\pi)\\ &= \Lambda(\xi, \mathrm{Ad}_h \pi) = \beta(\xi)_h,\end{aligned}$$

so $\beta(\xi)$ is right-invariant. Morever, if $\eta_R(h) = L_{h*}\eta$ is the fundamental vector field associated to the right action corresponding to η, then $\theta(\eta_R) = \eta$. Hence $\pi(\eta_R) = 0$ and

$$\beta(\xi)_h(\eta_R) = 0.$$

$\beta(\xi)$ is also right-horizontal, hence right-basic. This proves the first claim of the proposition.

For the second, note that θ and hence π are left H-invariant, so

$$\begin{aligned}\left(L_{g^{-1}}^*\beta(\xi)\right)_h &= \Lambda(\xi, Ad_{g^{-1}h}\pi)\\ &= \Lambda(\mathrm{Ad}_g \xi, \mathrm{Ad}_h \pi) = \beta(\mathrm{Ad}_g \xi)_h.\end{aligned}$$

Finally, to prove the third claim, we will show that for $\xi \in \mathfrak{g}$,

$$\iota(\xi_L)\beta(\xi) = 0. \tag{32}$$

Indeed,

$$\begin{aligned}\beta(\xi)_h(\xi_L) &= \Lambda\left(\xi, \mathrm{Ad}_h \frac{\mathrm{Ad}_{h^{-1}} - \mathrm{Ad}_{\sigma(h^{-1})}}{2}\xi\right)\\ &= \frac{1}{2}\Lambda(\xi,\xi) - \frac{1}{2}\Lambda(\mathrm{Ad}_{h^{-1}}\xi, \mathrm{Ad}_{\sigma(h^{-1})}\xi) \qquad (33)\\ &= 0. \qquad (34)\end{aligned}$$

Here the last step is from the s-skewness of Λ. The proposition is proved. □

Remark. It is only in (33) that we used the full Ad_H-invariance of the pairing Λ. In fact, the first two claims of Proposition 3 can be proven with only a pairing between $\mathfrak{g}$ and $\mathfrak{p}$ which is Ad_G-invariant (note Ad_G preserves the decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$). There is a unique extension of such a pairing to an s-skew pairing of the full Lie algebra, and to force the associated one-form to obey (32) is essentially the assumption that this extension is completely invariant.

2.4. **The invariant form.** To summarize the last subsection, we have found an equvariant three-form $\tau \in \Omega^1(H/G)$ that has the hope of being part of a closed form. We now complete this process by exhibiting the invariant piece.

Proposition 4. *Define $\Upsilon \in \Omega^3(H)$ by*

$$\Upsilon = \frac{1}{3}\Lambda(\pi, [\pi,\pi]). \tag{35}$$

Then

(a) *Υ is right G-basic and left G-invariant. Hence there exists a unique $\Xi \in \Omega^3(P)^G$ such that $\Upsilon = j^*\Xi$.*

(b)

$$d\Xi = 0. \tag{36}$$

(c) *For $\xi \in \mathfrak{g}$,*

$$i_\xi \Xi = d\tau(\xi). \tag{37}$$

We postpone the proof for a quick lemma.

Lemma 2. If $\theta = \gamma + \pi$ is the decomposition of θ relative to that of $\mathfrak{h}$, then

$$\begin{aligned} d\gamma &= -\frac{1}{2}([\gamma,\gamma] + [\pi,\pi]); \\ d\pi &= -[\gamma,\pi]. \end{aligned}$$

Proof. This is an immediate consequence of the bracket identities for a symmetric Lie algebra and the Cartan structure equations. □

Proof of Proposition 4.

(a) This is proved similarly to the analogous claim in Proposition 3.

(b) We are helped symbolically by the summation conventions on θ, γ and π. Note that by the Jacobi Identity

$$\big[\pi,[\pi,\pi]\big] = \big[\theta,[\theta,\theta]\big] = \big[\gamma,[\gamma,\gamma]\big] = 0$$

Thus,

$$\begin{aligned} d\Upsilon &= \frac{1}{3} d\Lambda(\pi,[\pi,\pi]) \\ &= \Lambda(d\pi,[\pi,\pi]) \\ &= -\Lambda([\gamma,\pi],[\pi,\pi]) \\ &= -\Lambda\left(\gamma,\big[\pi,[\pi,\pi]\big]\right) = 0, \end{aligned}$$

so (36) is proved.

(c) Let $\xi \in \mathfrak{g}$. Then

$$\begin{aligned} \iota(\xi_L)\Upsilon &= \frac{1}{3}\iota(\xi_L)\Lambda(\pi,[\pi,\pi]) \\ &= \Lambda\left(\pi(\xi_L),[\pi,\pi]\right) \\ &= \Lambda(Ad_{h^{-1}}\xi,[\pi,\pi]). \end{aligned}$$

On the other hand,

$$\begin{aligned} d\beta(\xi) &= d\Lambda(\xi, \mathrm{Ad}_h \pi) \\ &= \Lambda(\xi, \mathrm{Ad}_h \mathrm{ad}_\theta \pi) - \Lambda(\xi, \mathrm{Ad}_h[\pi,\gamma]) \\ &= \Lambda(\xi, \mathrm{Ad}_h[\gamma+\pi,\pi]) - \Lambda(\xi, \mathrm{Ad}_h[\pi,\gamma]) \\ &= \Lambda(\xi, \mathrm{Ad}_h[\pi,\pi]). \end{aligned}$$

Thus (37) is true as well.

□

As an immediate consequence, we have:

Theorem 1. *If (H, G) is a Legendrian symmetric pair, the equivariant three-form $\tilde{\Xi} = \Xi + \tau$ is equivariantly closed, thus giving $P = H/G$ the structure of a premoment space for G.* $\square$

2.5. **Nondegeneracy.** Along with the equivariant condition (7), which we have just satisifed for an arbitrary Legendrian symmetric pair, there is the nondegeneracy (actually, minimal degeneracy) condition (8). This condition is nearly independent of the results preceding, but becomes essential when actually working with P-Hamiltonian spaces (note that any G-space with equivariant form 0 is a premoment space for G). Here we will use the nondegeneracy of the pairing to satisfy nondegeneracy of τ.

Proposition 5. *Let (H, G) be a Legendrian symmetric space, and $\mathcal{O}$ an orbit of G in $P = H/G$. Then $\mathcal{O}$ with two-form given by (11) and moment map $i\colon O \to P$ satifies*

$$\ker \omega_p = \left\{ \xi_P(p) \;\middle|\; \xi \in \ker \tau_p \colon \mathfrak{g} \to T_p^* P \right\}. \tag{38}$$

Hence $\mathcal{O}$ is a P-Hamiltonian G-space.

Again, the theorem follows immediately:

Theorem 2. *Let (H, G) be a Legendrian symmetric pair. Then $P = H/G$ is a moment space for G.* $\square$

Proof of Proposition 5. What we are attempting to prove is

$$\iota_\xi \omega = 0 \iff \begin{cases} \xi_P = 0 & \text{or} \\ \xi \in \ker \tau_p \end{cases} \tag{39}$$

Suppose that $\xi \in \ker \tau_p$, where $p = hG$. This means that

$$0 = \Lambda \left(\operatorname{Ad}_{h^{-1}} \xi, \frac{1-s}{2} \operatorname{Ad}_{h^{-1}} \eta \right) = -\frac{1}{2} (\operatorname{Ad}_{h^{-1}} \xi, \operatorname{Ad}_{\sigma(h^{-1})} \eta)$$

for all $\eta \in \mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{g}^\Lambda$, we have that

$$\operatorname{Ad}_{\sigma(h)h^{-1}} \xi \in \mathfrak{g}. \tag{40}$$

Write $k = \sigma(h)h^{-1}$ and note that $\sigma(k) = k^{-1}$. Then by (40) we must have

$$\operatorname{Ad}_{k^{-1}} \xi = \sigma \left(\operatorname{Ad}_k \xi \right) = \operatorname{Ad}_k \xi$$

and therefore $\xi = \operatorname{Ad}_{k^2} \xi$ or

$$\xi \in \ker(1 - \operatorname{Ad}_{k^2}).$$

Now we have a direct-sum decomposition

$$\ker(1 - \operatorname{Ad}_{k^2}) = \ker(1 - \operatorname{Ad}_k) \oplus \ker(1 + \operatorname{Ad}_k).$$

If ξ is in the first summand, we have $\operatorname{Ad}_{h^{-1}} \xi = \operatorname{Ad}_{\sigma(h^{-1})} \xi \in \mathfrak{g}$ and therefore $\xi_P(p) = 0$. On the other hand, if ξ is in the second summand we have

$$\operatorname{Ad}_{h^{-1}} \xi = -\operatorname{Ad}_{\sigma(h^{-1})} \xi \in \mathfrak{p}$$

and thus $\beta(\xi)_h = 0$. Therefore (39) is true. $\square$

3. A Decomposition Theorem

We have shown how Legendrian symmetric pairs can give moment spaces. We now attempt to show the extent to which the known examples are the only ones.

Of course, if $G = G_1 \times G_2$ is a direct product of Lie groups, and P_1 and P_2 are moment spaces for G_1 and G_2, respectively, then $P_1 \times P_2$ with equivariant form $\tilde{\Xi}_1 + \tilde{\Xi}_2$ is a moment space for G. Thus we have a way of "building up" moment spaces. It is natural to try to go the other way—i.e., to decompose.

Theorem 3. *Let $(\mathfrak{h}, s)$ be an effective, orthogonal symmetric Lie algebra with Legendrian structure Λ. Then there exists a unique canonical (up to isometry) decomposition*

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-; \qquad \textit{(direct sum of ideals)} \tag{41}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-; \qquad \textit{(direct sum of ideals)} \tag{42}$$

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \qquad \textit{(direct sum of subspaces)} \tag{43}$$

such that, with the induced symmetric and Legendrian structures given by restriction

$$\mathfrak{h}_0 = \mathfrak{g}_0 \oplus \mathfrak{p}_0 \cong \mathfrak{g}_0 \rtimes \mathfrak{g}_0^*;$$
$$\mathfrak{h}_+ = \mathfrak{g}_+ \oplus \mathfrak{p}_+ \cong \mathfrak{g}_+ \times \mathfrak{g}_+;$$
$$\mathfrak{h}_- = \mathfrak{g}_- \oplus \mathfrak{p}_- \cong \mathfrak{g}_- \otimes \mathbb{C}.$$

Remark. This is very similar to the decompostion of effective, orthogonal symmetric Lie algebras into a sum of Euclidean, compact, and noncompact pieces given in [8, Ch. V, Theorem 1.1].

We will prove this in a sequence of lemmata. To begin with, let B be the negative of the Killing form on $\mathfrak{h}$. Then B is positive-definite on $\mathfrak{g}$, $\mathrm{ad}_{\mathfrak{h}}$-invariant, and also s-invariant:

$$B(s\zeta_1, s\zeta_2) = B(\zeta_1, \zeta_2),$$

or,

$$B(s\zeta_1, \zeta_2) = B(\zeta_1, s\zeta_2),$$

for all ζ_1, $\zeta_2 \in \mathfrak{h}$. Define $J\colon \mathfrak{h} \to \mathfrak{h}$ by

$$\Lambda(J\zeta_1, \zeta_2) = B(\zeta_1, \zeta_2). \tag{44}$$

Lemma 3.

(a) J commutes with the adjoint action of $\mathfrak{h}$ on itself. That is, for all $\zeta \in \mathfrak{h}$,

$$J \circ \mathrm{ad}_\zeta = \mathrm{ad}_\zeta \circ J;$$

or, for all ζ_1 and ζ_2,

$$J[\zeta_1, \zeta_2] = [J\zeta_1, \zeta_2];$$

(b) J anti-commutes with s: $J \circ s = -s \circ J$. So J takes $\mathfrak{g}$ into $\mathfrak{p}$ and *vice-versa.*
(c) J is self-adjoint with respect to Λ;.
(d) $J|_{\mathfrak{g}}$ is a vector-space isomorphism $\mathfrak{g} \cong \mathfrak{p}$.

Proof. Parts (a) and (b) are straightforward. Part (c) is a simple consequence of symmetry of B and Λ. Part (d) follows from the fact that B is positive definite on $\mathfrak{g}$. □

It follows that J^2 is an endomorphism of $\mathfrak{g}$. By Lemma 3, Part (c), J^2 is self-adjoint. Therefore, $\mathfrak{g}$ has an orthonormal (with respect to B) basis of eigenvectors with real eigenvalues. By rescaling Λ by positive constants, we may assume all nonzero eigenvalues are ± 1. Put

$$\begin{aligned} \mathfrak{g}_0 &= \ker J^2; \\ \mathfrak{g}_+ &= +1 \text{ eigenspace of } J^2; \\ \mathfrak{g}_- &= -1 \text{ eigenspace of } J^2; \end{aligned}$$

Lemma 4. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is a direct sum decomposition of ideals orthogonal with respect to B. In fact,

$$[\mathfrak{g}_0, \mathfrak{g}_+] = [\mathfrak{g}_0, \mathfrak{g}_-] = [\mathfrak{g}_-, \mathfrak{g}_+] = 0. \tag{45}$$

Proof. Let ξ, $\eta \in \mathfrak{g}$ have eigenvalues α and β, respectively. Then

$$\alpha B(\xi, \eta) = B(J^2\xi, \eta) = B(\xi, J^2\eta) = \beta B(\xi, \eta).$$

Hence $\alpha = \beta$ or $B(\xi, \eta) = 0$. Also,

$$\alpha[\xi, \eta] = [J^2\xi, \eta] = J^2[\xi, \eta] = [\xi, J^2\eta] = \beta[\xi, \eta],$$

so the summands are ideals, and the commutation relations (45) hold. □

Now let

$$\begin{aligned} \mathfrak{p}_0 &= J\mathfrak{g}_0 & \mathfrak{h}_0 &= \mathfrak{g}_0 \oplus \mathfrak{p}_0; \\ \mathfrak{p}_+ &= J\mathfrak{g}_+ & \mathfrak{h}_+ &= \mathfrak{g}_+ \oplus \mathfrak{p}_+; \\ \mathfrak{p}_- &= J\mathfrak{g}_- & \mathfrak{h}_- &= \mathfrak{g}_- \oplus \mathfrak{p}_-; \end{aligned}$$

and note that (43) and (41) hold *modulo* the proof that the summands of $\mathfrak{h}$ are ideals. We prove this next.

Lemma 5. The following commutation relations hold:

$$[\mathfrak{p}_0, \mathfrak{p}_+] = [\mathfrak{p}_0, \mathfrak{p}_-] = [\mathfrak{p}_-, \mathfrak{p}_+] = 0. \tag{46}$$

Furthermore,

$$[\mathfrak{p}_0, \mathfrak{p}_0] = 0; \quad [\mathfrak{p}_+, \mathfrak{p}_+] \subset \mathfrak{g}_+; \quad [\mathfrak{p}_-, \mathfrak{p}_-] \subset \mathfrak{g}_-.$$

Proof. Any element of $\mathfrak{p}$ can be written as $J\xi$ for $\xi \in \mathfrak{g}$. Let ξ have J^2-eigenvalue α and η have eigenvalue β. Then since

$$[J\xi, J\eta] = J^2[\xi, \eta] = \alpha[\xi, \eta] = \beta[\xi, \eta],$$

we can apply Lemma 4 and get the desired result. □

All we need to do now is supply the isomorphisms. For $v \in \mathfrak{p}_0$, there is a unique covector $\iota_\eta \Lambda \in \mathfrak{g}_0^*$ (that is, Λ induces an isomorphism $\mathfrak{p}_0 \cong \mathfrak{g}_0^*$). Define

$$\begin{aligned} T_0 \colon \mathfrak{h}_0 &\longrightarrow \mathfrak{g}_0 \rtimes \mathfrak{g}_0^* \\ (\xi, v) &\longmapsto \xi \rtimes \iota_\eta \Lambda; \end{aligned}$$

T_0 is obviously bijective. Since $[\mathfrak{p}_0, \mathfrak{p}_0] = 0$, it follows that T_0 is a homomorphism of symmetric Lie algebras, and thus an isomorphism.

Define another bijection

$$\begin{aligned} T_+ \colon \mathfrak{h}_+ &\longrightarrow \mathfrak{g}_+ \times \mathfrak{g}_+ \\ (\xi, J\eta) &\longmapsto \frac{1}{2}(\xi + \eta, \xi - \eta); \end{aligned}$$

Given the fact that $[J\xi, J\eta] = [\xi, \eta]$ on this subalgebra, T_+ is an isomorphism. Finally, define

$$\begin{aligned} T_- \colon \mathfrak{h}_- &\longrightarrow \mathfrak{g}_- \otimes \mathbb{C}; \\ (\xi, J\eta) &\longmapsto \xi + \sqrt{-1}\eta; \end{aligned}$$

Since this time $[J\xi, J\eta] = -[\xi, \eta]$, T_- is also an isomorphism. This proves the theorem.

Theorem 4. *Let (H, G) be a Legendrian symmetric pair, with G connected and semisimple, and H simply connected. Then the moment space $P = H/G$ has a decomposition*

$$P = P_0 \times P_+ \times P_-$$

and G has a decomposition

$$G = G_0 \times G_+ \times G_-$$

such that P_0 is a moment space for G_0 isomorphic to $\mathfrak{g}_0^*$, P_+ is a moment space for G_+ isomorphic to G_+, and P_- is a moment space for G_- isomorphic to $P^{\mathbb{C}}(G_-)$.

Proof. Cf. [8, Ch. V, Prop. 4.2] It follows from the homotopy exact sequence for the fibration $G \to H \to H/G$ that if H is simply connected and G is connected, then H/G is simply connected.[4]

Since G is semisimple, $\mathfrak{h}$ is an effective, orthogonal, Legendrian, symmetric Lie algebra Therefore, we can compose $\mathfrak{h}$ as in Theorem 3 into $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-$. Let $H_0 \times H_+ \times H_-$ be the corresponding decomposition of H. Likewise $\mathfrak{g}$ decomposes and we can write $G = G_0 \times G_+ \times G_-$. Then

$$P = H/G = \frac{H_0 \times H_+ \times H_-}{G_0 \times G_+ \times G_-} = H_0/G_0 \times H_+/G_+ \times H_-/G_-.$$

□

4. Conclusions

Of course, we have said nothing about further axioms of a moment space we might want to assume, in order to find the correct analogues of fusion and reduction, etc. It is quite possible that with the addition of certain axioms, a moment space *must* be a globally symmetric space, and therefore must have an associated symmetric pair. This would complete the classification of moment spaces.

[4] Recall that for a topological group G, π_0 has a group structure. In fact, $\pi_0(G) = G/G_0$, where G_0 is the connected component of the identity. Thus the exact sequence is one of groups even on the π_0 level.

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